Soviet Physics Doklady 1960 Vol. 4 No. 6 pp. 1210–1213.

Fluid Mechanics

THE BOUNDED VARIATION OF CONTINUOUS SOLUTIONS OF THE HYDRODYNAMIC EQUATIONS

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(Presented by Academician M. V. Kelbyshii, August 15, 1959)

(Translated from: Doklady Akad. Nauk SSSR Vol. 129, No. 6, pp. 1257-1260, November-December, 1959)

(Original article submitted August 11, 1959)

The equations of hydrodynamics have the form

$$\frac{\partial u^\alpha}{\partial t} + \frac{\partial U^\alpha}{\partial x} = 0, \tag{1}$$

where (if Lagrangian coordinates are used),

$$\begin{aligned} u^{(1)} &= u, \quad u^{(2)} = v, \quad u^{(3)} = E + \frac{u^2}{2}, \\ U^{(1)} &= p, \quad U^{(2)} = -u, \quad U^{(3)} = pu, \end{aligned} \tag{2}$$

with u the velocity, v the specific volume, p the pressure, $E = E(p, v)$ the specific internal energy.

Our aim in this paper is to prove that the solution of (1) is of bounded variation. In order to do this, we assume, first of all, that the initial values are of bounded variation and, secondly, that the solution is continuous in the strip $0 \leq t \leq t_0$, $-\infty < x < +\infty$. The second assumption is not essential, but it makes it possible to carry out the proof by using a relatively uncomplicated technique. It is found that important features of the system of hydrodynamic equations appear when these equations are taken as special cases of the hyperbolic system

$$\frac{\partial u^\alpha}{\partial t} + U^\alpha_\beta \frac{\partial u^\beta}{\partial x} = 0 \tag{3}$$

(where U^α_β are functions of the variables u^α).

The hydrodynamic types of system are distinguished by the presence in them of functions which, in a certain relation, play a part similar to that of entropy in the hydrodynamic equations. Since this relation is in the form of an equality, such "entropic" systems form a very small class of the general hyperbolic systems. The proof that the solutions are of bounded variation is given below for all these systems. "Entropic" systems occupy an intermediate position between linear systems like the Cauchy type of problem, and general systems (3) for which the solutions, in general, tend to infinity for finite values of t even with initial values that are constant outside a certain interval, and smooth inside it.

We will now consider the method of proof. The calculations gain considerably in clarity as they are carried out in the general form for a system (3) with an arbitrary number of equations. Our first goal is to rewrite the system (3) in the form of an equivalent system of integral equations, from which we can then obtain an estimate of $\int_{-\infty}^{+\infty} \left| \frac{\partial u^\alpha}{\partial x} \right| dx$. This can be done by doubling the number of unknown functions by introducing, in addition to the u^α, the functions $v^\alpha = \partial u^\alpha / \partial x$. The system of equations satisfied by v^α is obtained by differentiating the equations of (3) with respect to x, and has the form:

$$\frac{\partial v^\alpha}{\partial t} + \frac{\partial}{\partial x}(U^\alpha_\beta v_\beta) = 0. \tag{4}$$

In order to obtain the integral equations desired, it is convenient to replace the system (4) by a simpler diagonalized system for the new set of functions φ^μ, which are linear combinations of the v^α with coefficients depending on u^α:

$$\varphi^\mu = l^\mu_\alpha v^\alpha. \tag{5}$$

The functions φ^μ are no longer the total derivatives of any function of u^α as the v^α were. Nevertheless, $\varphi^\mu dx = l^\mu_\alpha du^\alpha$ is a linear combination of the differentials du^α, and it is convenient to denote this expression by $\delta\psi^\mu = l^\mu_\alpha du^\alpha$ as is done in thermodynamics, and to speak of a special "differential," remembering, of course, that we cannot always find a function $\psi^\mu [u^{(1)}, u^{(2)}, \ldots, u^{(n)}]$ having $l^\mu_\alpha du^\alpha$ as a differential. The matrix inverte to l^μ_α will be written as m^α_ν, so that $l^\mu_\alpha m^\alpha_\nu = \delta^\mu_\nu$. After a relatively simple calculation, we obtain:

$$\frac{\partial \varphi^\mu}{\partial t} + \Lambda^\mu_\alpha \frac{\partial \varphi^\alpha}{\partial x} + \frac{\partial \Lambda^\mu_\alpha}{\partial x}\varphi^\alpha = C^\mu_{\alpha\beta}\varphi^\alpha\varphi^\beta, \tag{6}$$

where

$$\Lambda^\mu_\nu = l^\mu_\gamma U^\gamma_\delta m^\delta_\nu, \tag{7}$$

$$C^\mu_{\alpha\beta} = \left(\frac{\partial l^\mu_\gamma}{\partial u^\delta} - \frac{\partial l^\mu_\delta}{\partial u^\gamma}\right)\Lambda^\gamma_\varepsilon m^\delta_\alpha m^\varepsilon_\beta. \tag{8}$$

The existence of a matrix l^μ_γ which transforms U^γ_δ into diagonal form Λ^μ_ν follows, of course, directly from the fact that the system (3) is hyperbolic. The matrix m^δ_ν is made up simply of the eigenvectors of U^γ_δ, while l^μ_γ is formed from the set of vectors biorthogonal to these eigenvectors. Since the eigenvectors are defined only to within an arbitrary multiplying factor, however, the matrix l^μ_γ can be multiplied on the left by an arbitrary diagonal matrix without changing Λ^μ_ν.

The fact that we have this freedom of choice can be used to simplify the tensor $C^\mu_{\alpha\beta}$ as much as possible. We will try, for example, to reduce the right-hand side of one of the equations of the system (6) to zero. It is evident from (8) that, in general (for $\Lambda^\mu_\nu \neq 0$), this will occur when, and only when

$$\frac{\partial l^\mu_\gamma}{\partial u^\delta} - \frac{\partial l^\mu_\delta}{\partial u^\gamma} = 0, \tag{9}$$

i.e., if, and only if, the special "differential" $\delta\psi^\mu = l^\mu_\alpha du^\alpha$ is a total differential. In more precise terms, it will occur when the special differential has an integrating factor. For a system of two equations, both the special differentials have integrating factors. For systems of more equations, none of the special differentials will, as a rule, have an integrating factor. Thus, it is particularly striking that the hydrodynamic equations are exceptions to this rule. The existence of an integrating factor for one of the special "differentials" arising from this system of equations is equivalent to the known thermodynamic identity

$$dS = \frac{1}{T}(dE + p\,dv). \tag{10}$$

This remark forms the basis for introducing the term "entropy" for any integral special "differential."

We now consider the general system (6), and note that the solutions of such a system tend to infinity for a finite value t_0, no matter how smooth the function defining the initial condition for $t = 0$. This is well illustrated by the behavior of the model* system

$$\frac{\partial\varphi^{(1)}}{\partial t} + \frac{\partial\varphi^{(1)}}{\partial x} = \varphi^{(2)}\varphi^{(3)}, \qquad \frac{\partial\varphi^{(2)}}{\partial t} = \varphi^{(3)}\varphi^{(1)}, \qquad \frac{\partial\varphi^{(3)}}{\partial t} - \frac{\partial\varphi^{(3)}}{\partial x} = \varphi^{(1)}\varphi^{(2)} \tag{11}$$

with initial conditions $\varphi^{(1)}(x,0) = \varphi^{(2)}(x,0) = \varphi^{(3)}(x,0) = a(x)$ for $t = 0$, where $a(x)$ is an even function of x equal to unity in the interval (0,2), smoothly decreasing to zero in (2,3), and equal to zero on the ray $(3, \infty)$. It is not difficult to verify that, in the rectangle $-1 \le x \le 1$, $0 \le t < 1$, the solution is given by the formula

$$\varphi^{(1)}(x, t) = \varphi^{(2)}(x, t) = \varphi^{(3)}(x, t) = \frac{1}{1-t}. \tag{12}$$

The behavior shown by the example of the system (11) is essentially different from the behavior of an "overturning front" (a "gradient catastrophe") in the case of a single equation (or of a system of two equations). The difference lies in the fact that, in the case of a single equation, the growth of φ automatically causes the characteristics to concentrate in a smaller region, and this compensates for the increase in φ. This is clearly evident from Eq. (6). If $C^{\mu}_{\alpha\beta} = 0$, then, for increasing φ, it is necessary that $\partial\Lambda/\partial x < 0$, and this means that the characteristics are being squeezed together. It is not difficult to verify that this "compression" of the characteristics is just sufficient to keep the integral of the absolute value of φ constant. If $C^{\mu}_{\alpha\beta} \neq 0$, it can happen that the terms of $C^{\mu}_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}$ cause a growth of φ which is not compensated by a compression of the characteristics. In fact, this can cause a partial spreading of the characteristics so that $\int_{-\infty}^{+\infty} |\varphi|\,dx$ is increased by both causes.

A separation of the terms of $C^{\mu}_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}$ thus means a separation of an uncompensated part of the quadratic form, and this has a strong effect in causing the solution to tend to infinity for a finite t_0. The equations (6) and relations (8) clearly show the difference in origin of terms of the type $\frac{\partial\Lambda^{\mu}_{\alpha}}{\partial x}\varphi^{\alpha} = \frac{\partial\Lambda^{\mu}_{\alpha}}{\partial u^{\gamma}} m^{\gamma}_{\beta}\varphi^{\alpha}\varphi^{\beta}$ and of the type $C^{\mu}_{\alpha\beta}\varphi^{\alpha}\varphi^{\beta}$. The first have their origin with the variability of the eigenvalues of the matrix U^{α}_{β}, and their effect is automatically compensated by a compression of the characteristics. The second are produced by the rotation** of the eigenvectors, and for them there are no obvious compensating factors. In each case, these factors are absent if, in the system (6), the coefficients depending on u are replaced by constants.

We now turn to the proof of the basic theorem, and, first of all, give a precise definition of a hydrodynamic-type system of equations.

Definition 1. A special "differential" $\delta\psi^{\mu} = l^{\mu}_{\alpha}du^{\alpha}$ and the corresponding equation in (6) are called entropic, if $\delta\psi^{\mu}$ is a total differential, i.e., if $C^{\mu}_{\alpha\beta} = 0$ for all α and β.

Definition 2. The system of equations (3) is called a hydrodynamic-type system if it has at least one

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*This system is typical in the sense that it is not clear that there is a system of form (3), much less a system of the form (1), from which the system (11) can be obtained, by using the above construction. It follows from (7) that a system of the type (3) with arbitrary constant Λ^{μ}_{ν} can be found. In this case, the freedom of choice of the matrix l^{μ}_{ν} may make it possible to obtain constant values for $C^{\mu}_{\alpha\beta}$. It nevertheless remains completely unknown whether such a method could lead to the construction of a system of the type (11).

**Something similar probably happens in the case of systems $dr/dt = A(t)r$ of ordinary equations. We set $A = U\Lambda U^{-1}$, where Λ is a constant triangular matrix with negative eigenvalues, and $U = e^{St}$, where S is a constant antisymmetric matrix. Even though all the eigenvalues of A are negative, according to Lyapunov it is possible to draw conclusions concerning the stability of the system only for slow rotations of U, i.e., for small S. If, for example, $\Lambda = \begin{pmatrix} -1 & 7 \\ 0 & -2 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then the system becomes unstable.

"entropic" equation, and if, in the nonentropic equations of the system (6) the coefficients $C^{\mu}_{\alpha\beta}$ of any two non-entropic terms $\varphi^{\alpha}\varphi^{\beta}$ are equal to zero.

Note. For a system of three equations to be of the hydrodynamic type, it is sufficient for it to have an "entropy". In this case, the matrix l^{μ}_{α} can be chosen so that the tensor $C^{\mu}_{\alpha\beta}$ will have the desired structure. To do this, it is sufficient to make two coefficients C^{1}_{12} and C^{2}_{12} equal to zero. The equations $C^{1}_{12} = 0$ and $C^{2}_{12} = 0$ yield two conditions, one on each of two nonentropic eigenvectors.

We will assume now that the system is of the hydrodynamic type, and will change the order of the equation in (6) so that the entropic equations come first:

$$\frac{\partial\varphi^{\mu}}{\partial t} + \frac{\partial}{\partial x}(\Lambda^{\mu}_{\alpha}\varphi^{\alpha}) = 0, \quad 1 \leqslant \mu \leqslant n_0; \tag{13}$$

$$\frac{\partial\varphi^{\nu}}{\partial t} + \frac{\partial}{\partial x}(\Lambda^{\nu}_{\alpha}\varphi^{\alpha}) = p^{\nu}_{\gamma}\varphi^{\gamma} + q^{\nu}, \quad n_0 + 1 \leqslant \nu \leqslant n. \tag{14}$$

Here the coefficients p^{ν}_{γ} are expressible linearly, and the coefficients q^{ν} quadratically in terms of the "entropic" terms φ^{μ}.

For a hydrodynamic-type system there is, therefore, a "splitting off" of the entropic equations, which (if it is assumed that the functions Λ^{μ}_{ν} and $C^{\mu}_{\alpha\beta}$ are given, which is allowable for obtaining a priori estimates) can be integrated independently, and which have solutions for which $\int\limits_{-\infty}^{+\infty} |\varphi^{\mu}(x, t)|\,dx$ is uniformly bounded for all $t > 0$.

The nonentropic terms form a system of linear equations with coefficients depending on the entropic terms. For this system, the integrals $\int\limits_{-\infty}^{+\infty} |\varphi^{\nu}(x, t)|\,dx$ are generally no longer bounded. Their rate of growth, however, is not greater than $\exp\left(\int\limits_{0}^{t} a\,dt\right)$, where

$$a(t) = \max_{1\leqslant\mu\leqslant n_0} \int\limits_{-\infty}^{+\infty} |\varphi^{\mu}(x, t)|\,dx.$$

This result can be proven by writing Eq. (14) as integral equations along the corresponding characteristics, and applying the method of successive approximations to the integral equations obtained.

Note. In this proof, it is essential to use the fact that $C^{\mu}_{\alpha\alpha} = 0$ for any system and for arbitrary μ and α. This result can be obtained directly from (8) by noting that Λ^{γ}_{ξ} is a diagonal matrix, and it leads to the simple estimate

$$\int\limits_{0}^{t_0}\int\limits_{-\infty}^{+\infty} |q^{\nu}(x, t)|\,dx\,dt \leqslant \text{const} \max_{\substack{1\leqslant\mu\leqslant n_0 \\ 0\leqslant t\leqslant t_0}} \left[\int\limits_{-\infty}^{+\infty} |\varphi^{\mu}(x, t)|\,dt\right]^2$$

for the terms q^{ν}.